The Rectangular Doorway to Algebra

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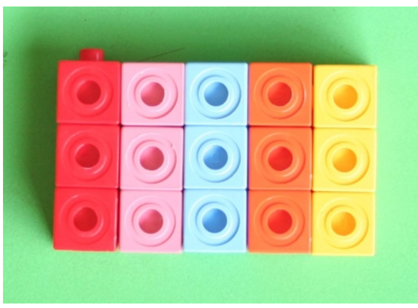
Abstract :

W.W. Sawyer in his "Vision in Elementary Mathematics" provides key insights for demystifying a child's first introduction to Algebra. Introducing 'x', through the simple game "Think of a number", teaching multiplication via area of rectangles, and building on this to understand multiplication of algebraic expressions and polynomials are the essence of Sawyer's approach . In addition "Vision" provides gems of pedagogical wisdom, such as "not introducing negative numbers in algebra initially", which facilitate understanding of important and useful aspects of algebraic multiplication.

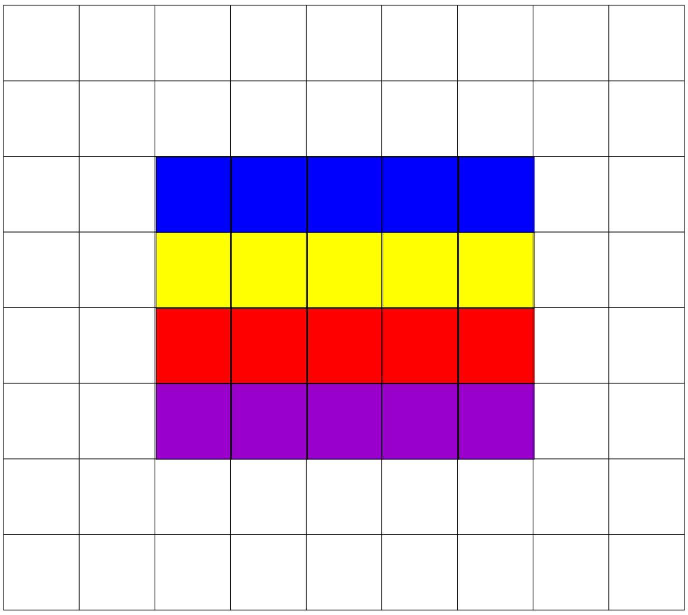
Sawyer's rectangular doorway to Algebra is put together with the two step learning process of Universal Active Mathematics- first learning by doing, by constructing physical shapes and sizes, and then translating this understanding into the language of pencil and paper, to build a concept construction sequence for introductory algebra. Place value multiplication and multiplication of polynomials in one variable are seen to be isomorphic, so that algebra at many levels is not a new subject. The new aspects of algebra arise in the solution of equations.

- Addition is joining : A good deal of all mathematics - and almost certainly, all of primary mathematics - can be constructed on the single principle that "addition is joining, joining is addition". Children learn addition readily with jodo blocks. Jodo blocks also translate effortlessly into exercises in grid books, with coloured squares representing blocks.

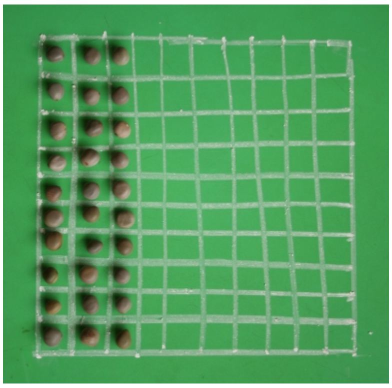
- Multiplication is rectangles : Multiplication is when we add the same number repeatedly to itself. With jodo blocks we can join the same number to itself repeatedly lengthwise to show the addition. But we can also join the number to itself repeatedly sideways. In this case what takes shape is a rectangle. When we join any number repeatedly to itself sideways, we always get a rectangle. So we can deduce a new principle "Multiplication gives rectangles, rectangles represent multiplication". This is not a new principle. It follows from our basic principle that "addition is joining". We can join the rods 'sideways' , left - right, or upwards and downwards. In each case we get the same rectangle but in different positions.

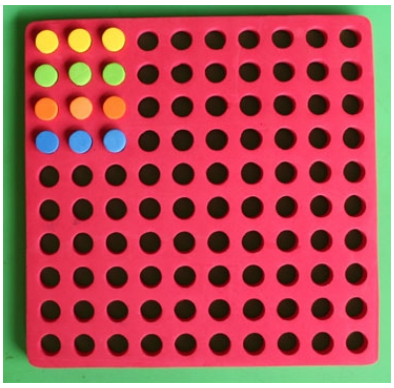


- Translation into a grid book: Rectangles assembled with cubes readily translate into rectangles drawn in the grid book . For example fig 1 represents the multiplication 4 x 5.



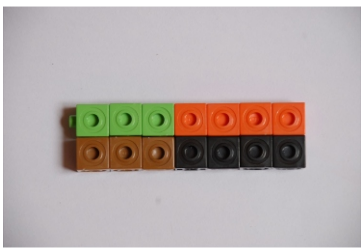
- Tables upto ten : Children can thus construct the multiplication tables as rectangles in the grid book, or as plugs on a mathemat, and then count and write the results. Note that when children are doing this they are using four different representations for the multiplication : the verbal four times five, the things representation four lines of plugs, five plugs in each line, the grid book diagram, and the alphanumeric 4 x 5. We may ask them to give a real life example of 4 times 5. When children do this, this is yet another, a fifth representation.

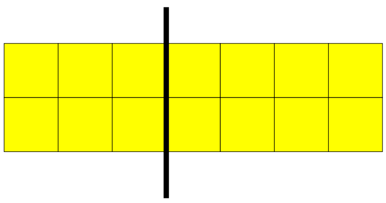




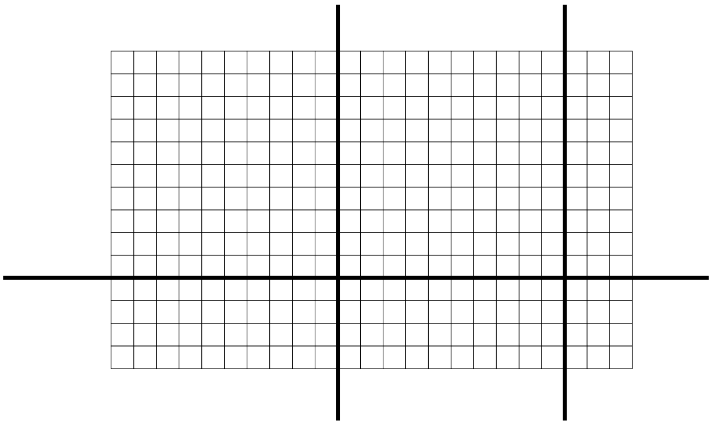
- The distributive law and rectangles : We can cut a rectangle in a grid book into two rectangles with a vertical cut. The original rectangle is now constructed as two smaller rectangles which are joined together . Since addition is joining, we can rewrite this multiplication by using brackets, and what we obtain is the distributive law.

$$2 \times 7 = 2 \times (3 + 4) = 2 \times 3 + 2 \times 4.$$

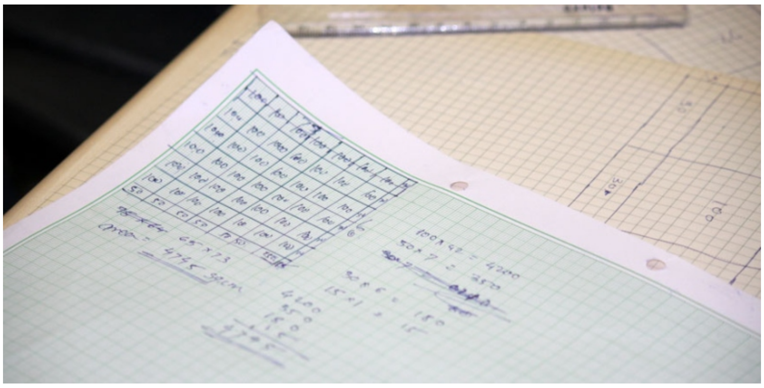




- Multiplication of two digit numbers : We do not need to know multiplication table beyond the tables of ten. We can multiply a two digit number like 23 by another like 14, by dividing the big rectangle into smaller rectangles. Of course, after they do a few examples, children discover that its easiest if we divide 23 into 10 + 10 + 3 etc. This is how children learn 2 digit multiplication in the NCERT textbook.



- Multiplication with Graph Paper : From grid book, we can transit to making rectangles on graph paper. With Standard graph paper, which has 1cmx 1cm squares and 1mm x 1mm squares, one can do larger 2 digit by 2 digit multiplications and even three digit multiplications. Such experimentation with rectangular representation of multiplication on graph paper readily leads to the discovery that, since ten times ten is hundred, it is easiest to read the answer by counting the 1 cm x 1 cm squares of hundred small squares each. They also discover that 100 times 100 is ten thousand, also a square on the same graph paper. They learn to recognize the 'rod' of ten, the squares of 100s, the larger rods of 1000s and the large square of 10,000.



- Translation into a plain white paper picture : The next, crucial step is ask the children to turn the graph paper over and draw a diagram for each multiplication that they have done on the graph paper. What they start drawing is something similar to this figure from “Vision in Elementary Mathematics” by WW Sawyer.

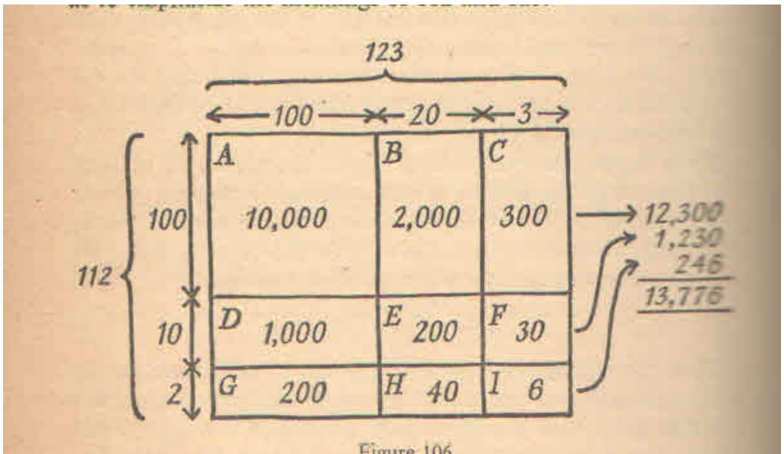


Figure 106

- Each translation into a new representation, such as from blocks to grid book, from grid book to graph paper, from graph paper to plain paper- is another step along the “ladder of conceptual construction', or the “concept construct sequence”. The important thing is that this is not something 'new' that the child has to accept. It is only something that has been constructed from the earlier components, which themselves were constructed from simpler components. Children while they are constructing the rectangles are also constructing conceptual structures for multidigit multiplication.

- Translation into the language of place value : Translating from the graphical representation of multidigit multiplication to the alphanumeric representation is done in a number of steps. We have already written the numbers 112 and 123 in the above figure as 100 + 10 + 3 , and 100 + 20 + 3. This is the standard 'expanded form'.

- We can also write 112 and 123 as 1 H + 1 T + 2, and 1 H + 2 T + 3.
  When the plain paper diagrams are re-labelled in this alphanumeric language, we notice that
  T x T = H, T x H = Th, and H x H = TTh

- From here it is a short step towards writing H as T x T, Th as TxTxT, and TTh as TxTxTxT. And from here it is another short step to H as T^2 and TTh as T^3 etc. Now we can write 123 as $1T^2 + 2T + 3$.

- Sawyer's introduction of unknown numbers : In 'Vision in Elementary Mathematics', Sawyer gives a simple and extremely effective way to effortlessly introduce the concept of an unknown number through playing the game “Think of a number”. This is best played in a classroom with all the children. They 'think of a number between one and ten', add three to it, then double the answer, add three to the answer, add the number that they had thought of to the answer, divide by three, subtract the number that they had thought of, and wonder of wonders, everyone gets 3 as their answer.

- Adding and subtracting unknown numbers : After a few such exercises, teacher shows how she does it with paper cups and cubes. Doubling one cup and three cubes gives two cups and six cubes. Adding three cubes gives two cups and nine cubes… and so on. We discover that we can add and subtract both known (cubes) and unknown (cups) numbers in the same way: “ addition is putting together” and keeping the account. Of course, there are simple rules like : We must put the same number in each cup. In this introductory phase the number which is put in the cup is a positive integer.

- Making algebraic expressions : The translation from a physical combination of cups and cubes to an algebraic expression is mediated by an exercise of drawing the combination on paper. In Sawyer's 'Vision', the unknown number is drawn as a bag, which later becomes 'x'.

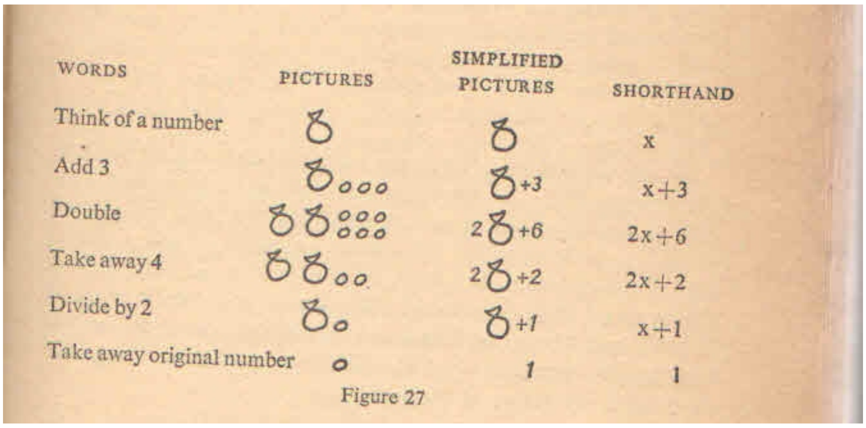
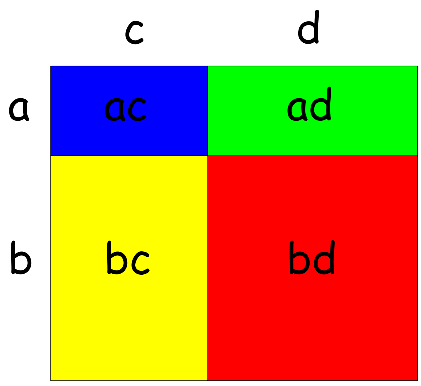


Figure 27

Three bags becomes 3x, and so on. At this stage we stay in the domain of positive integers. We write algebraic expressions like 5 x + 2. This is represented with 5 cups and 2 cubes. The value of the expression can be explored with different values for x. With this approach we can have two different colours of cups to represent two unknowns. The accounting principle does not change. Addition is joining. So 3X + 2Y + 3, when added to X + Y + 5 can be seen to be 4 X + 3 Y + 8.

- Multiplying unknown numbers : The next stage is to represent X times X. This can be done by joining two cups with a clothesline clip. X times Y, or X x Y, is a red paper cup clipped to a white paper cup.

- The distributive law with unknown numbers: Since "Multiplication is rectangles" leads to the distributive law when we cut the rectangle into smaller rectangles by vertical and horizontal cuts, we can now draw a rectangle representing (a + b) x (c + d).



Since now we are acquainted with multiplying two unknown numbers, adding (joining) the four component rectangles axc, axd, bxc, and bxd makes sense. This can be extended to (a + b)x (c + d + e) and so on. We discover that if there are 3 terms in the first bracket and 4 terms in the second bracket we get 3 x 4 terms in the result.

- Polynomials : Since now we know how to both add and multiply unknown numbers, we can construct representation of polynomials like $1 + 2X + 2X^2$ using cups and cubes.

- Algebra with integer non-negative coefficients: The 'number of terms' in the above polynomial is 5. Similarly, the number of terms in (a + 3b + 5c +d) is 10. When we multiply brackets, the number of terms in the answer is the product of the number of terms in each bracket. Thus the number of terms in $(a+b)^3$ is 2x2x2 = 8. Sure enough the coefficients, 1,3,3 and 1 on the right hand side add up to 8. That the number of terms must balance on each side when we do algebraic multiplication is not only fun to experiment with. It is also a quick way to check whether we have done the multiplication correctly.

- There is nothing new so far. What is new in algebra enters when we write equations. What is new is the concept of an inverse.

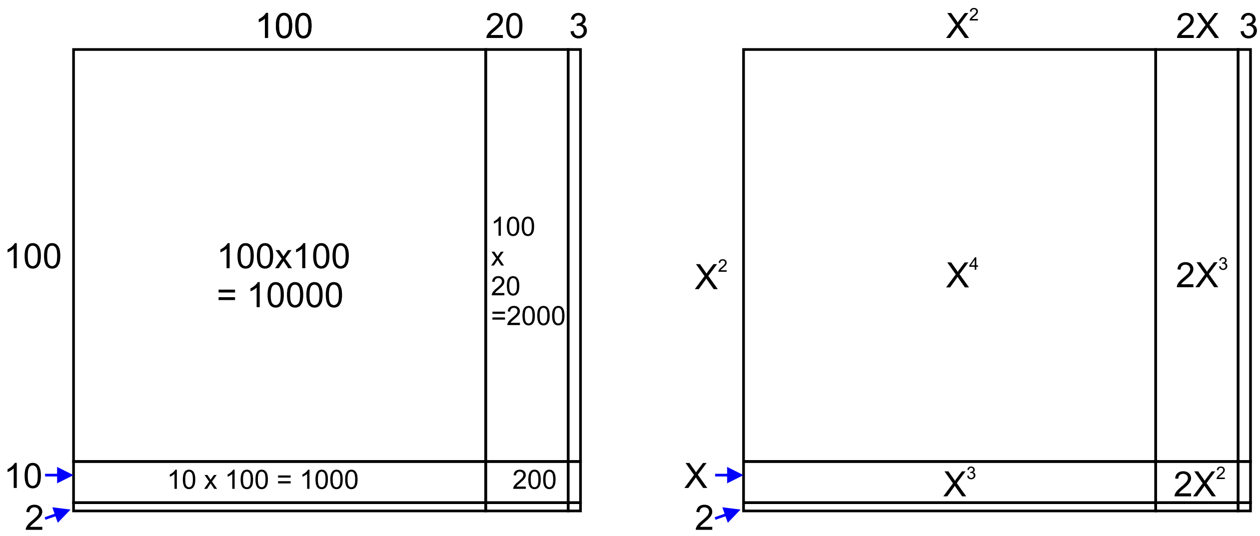
  We can set up an equation with cups and cubes. What's in the cup if two cups equals 6 cubes ? The answer, of course, is 3 cubes. What's in the cup if two cups equals 1 cube ? We get a new kind of number- a fractional number.

  We get negative numbers when we ask the question what's in the cup if one cup plus five cubes equals 2 cubes.
  i.e. we solve the equation X + 5= 2

We get irrational numbers when we solve the equation $X^2 = 2$
We get imaginary numbers when we solve $X^2 = -1$.

- **Algebra with negative integer coefficients -** Algebra with negative numbers requires a separate discussion which we do not enter into in this paper. Because with negative numbers we can have cancellation of terms, the numbers of terms on both sides of an equation may not be the same when we do algebra with negative coefficients.

- **Place value multiplication with negative integer coefficients :** Finally we note that when we multiply two polynomials like $(X^2 + X + 2)x(X^2 + 2X + 3)$ by drawing a rectangle, it is the same rectangular representation for multiplying 112 and 123. We just replace X by T.



- By convention, when we write numbers in place value notation we only use the positive integers from 1 to 9, and 0. Now since place value notation seems to be isomorphic to polynomials can we also write numbers in place value notation with negative coefficients ?
  Can we write 112 as $2T^2-9T+2$, and 123 as T^2+3T-7 ? Do we still get the same answer when we do the multiplication 112 x 123 ? Try it and see for yourself.

- In 'Vision', W W Sawyer writes : " In theory it has been recognized for years, indeed centuries, that children should understand and not merely parrot arithmetic. In practice, teaching for understanding is still far from universal. It therefore seems worth while to discuss ways of picturing the operations of arithmetic. These pictures will be helpful in learning arithmetic, and they are highly relevant to the beginnings of algebra." (W.W. Sawyer, Words, Signs, Pictures; Vision in Elementary Mathematics)
- "Algebra is concerned with the methods of arithmetic. When we teach arithmetic, whether we realize it or not, our real aim is to teach algebra. For it is extremely unlikely that our children will ever meet in later life the exact numbers they had in any problem at school. When we give them any particular exercise, our hope is that they will see that the same method could be applied to many similar problems. So that even when we are dealing with the *particular* numbers of arithmetic, we are hoping to convey the *general* ideas, which belong to algebra." (W.W. Sawyer, Words, Signs, Pictures; Vision in Elementary Mathematics)